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On existence and uniqueness of classical solutions to Euler equations in a rotating cylinder

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Abstract

In this paper we consider the existence and uniqueness of classical solutions of the non-stationary Euler equations for an ideal incompressible flow in a cylinder rotating with a constant angular velocity about an axis orthogonal to the axis of the cylinder. If we consider, say, one of the most-studied three-dimensional cases of fluid flow, axially-symmetric flow in a pipe, it can be easily seen that the (Navier–Stokes or Euler) equations are essentially two-dimensional because they split into a two-dimensional analog and a one-dimensional equation. The two-dimensional equations are known to have global classical solutions. If we add, as is standard in the literature, rotation along the axis of symmetry, the same conclusions are valid. However the picture completely changes if the axis of rotation does not coincide with the symmetry axis. The flow is then "stirred" and the splitting we mentioned above is coupled. Therefore in spite of the symmetry, i.e., the dependence on two spatial variables, the flow is essentially not two-dimensional. The flows of this type are called in physical literature *2D–3C flows* (two-dimensional, three-components flows). In the present paper we show that in this case for any $T > 0$ the unique classical solution exists on the interval $[0, T]$ for a sufficiently small angular velocity of rotation.

Keywords: Rotating fluids; Euler equations; Classical solution; Two-dimensional; Three-components flows

1. Introduction

The subject of rotating fluids has been attracting attention of researchers due to both practical and academic interest. Many interesting results were obtained which help the understanding of fluid dynamics. The study of rotational flow goes back to 1885 when H. Poincaré [1] obtained a system of equations by the linearization of the Euler equations about the solution which corresponds to rotation of the fluid as a rigid body around the z-axis. A systematic investigation of this new system of equations was initiated by S.L. Sobolev [2]. The case when the velocity and the pressure are dependent only on two spatial variables x and z and the fluid domain is an infinite cylinder with the axis parallel to the y axis was studied in detail in series of work pioneered by Alexandrian [3] and Zeleniak [4]. Fokin [5] recently showed that solutions of the Poincare–Sobolev equations in this configuration has a surprisingly complex vortex structure for a system of linear equations.

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The extension of Sobolev's analysis to the fully nonlinear Navier–Stokes and Euler equations was done by Babin, Mahalov and Nicolaenko in [6–9]. They studied the fluid flow with periodic boundary conditions for strong rotation. In particular they proved the global existence of regular solutions to the three-dimensional Navier–Stokes equations for sufficiently fast rotations. The global regular solution was shown for some classes of solutions on thin domains [10–13] where the flow is considered as a perturbed 2-D flow. Mahalov [14] proved the existence of global regular solutions for helical flows in pipes with "standard" *azimuthal* rotation.

At the same time, for essentially three-dimensional Euler equations there are no global existence results. Babin, Mahalov and Nicolaenko [9] proved the existence of regular solutions to Euler equations (with periodic boundary conditions) on a time interval which gets larger for faster rotation. Marsden, Ratiu and Raugel [15] produced an analogous result for thin shells with the interval depending on thickness of the shell. Mahalov, Nicolaenko, Bardos and Golse in their recent paper [16] proved non blow-up of the fully 3-D incompressible Euler equations for a class of three-dimensional initial data characterized by uniformly large vorticity in bounded cylindrical domains. The global regularity was proven without conditional assumptions on the properties of solutions at later times and the solutions were not close to some 2-D manifold in any sense.

In the present paper we consider Euler equations for an ideal flow in the same configuration as in Fokin [5].[1] The cylinder axis is perpendicular to the axis of rotation,[2] the angular velocity of rotation is constant, and the solution is axially symmetric. The flow in this case is essentially three-dimensional. This differs from "standard" rotation, when the axes of rotation and cylinder coincide, leading to the splitting of Euler equations into two- and one-dimensional equations.

We call a solution *a classical solution* if it, along with its derivatives that appear in the equations, is of class $C(\overline{Q}_T)$. The following result is proven for a cylinder with a simply-connected base, although with certain modifications, it can be proven for a multiply connected base using the same method employed in [17]. External forces are assumed to be absent. We show, for appropriate initial conditions, that the slower the rotating angular velocity of the cylinder, the longer a classical solution exists. If one considers the case when rotation and cylinder axes form the angle $\alpha \in [0, \pi/2]$, $\alpha = 0$ being the "standard" rotation and $\alpha = \pi/2$ being the case considered here, then one can prove existence of classical solution on a longer interval for a smaller α. This interval goes to ∞ as α goes to zero. The extension of the result to the case of arbitrary positioning of the cylinder and rotation axes and the presence of external force is straightforward along the lines of this paper and is merely a notational complication.

1.1. Problem

Let us consider our problem in a rotating Cartesian coordinate system (x, y, z), which is solidly connected to the cylinder. Let the y- and z-axes represent the rotation and cylinder axes respectively, so that the pressure and velocity components do not depend on the variable z. In this coordinate system, the problem is as follows: let $\Omega \subset \mathbf{R}^2(\mathbf{x}, \mathbf{y})$ be a simply-connected bounded domain with boundary $\partial\Omega$ of class $C^{2+\alpha}$. Then in $Q_T = \Omega \times [0, T]$ we have to find functions u, v, w, p satisfying the equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -p_x + \lambda w, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -p_y, \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = -\lambda u, \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

with the initial and boundary conditions

[1] Though the domain Ω does not have to be convex.

[2] The axes does not have to intercept. A long flow tube on the polar regions of Earth and an artesian shaft are good examples of the model under consideration (where axes are intercepted) as well as the same kind of flow located on the surface of Earth along latitude lines. As we comment later, the result is easily extended to arbitrary position of the "tube".

$$(u,v)|_{t=0}=(u_0,v_0),\quad (x,y)\in\Omega, \tag{5}$$

$$w|_{t=0}=w_0,\quad (x,y)\in\Omega, \tag{6}$$

$$(un_x+vn_y)|_{\partial\Omega}=0,\quad t\in[0,T], \tag{7}$$

where n_x, n_y are the components of the exterior normal to $\partial\Omega$ and $\lambda\in\mathbf{R}$ is the rotation angular velocity.

Without rotation ($\lambda=0$) the system of equations splits into subsystem (1), (2), (4), (5), (7) (which is the 2-D Euler equations and, as known, has a global classical solution, [17]) and into Eq. (3) with initial condition (6) (which can be solved in the usual way for known u and v).

For $\lambda\neq 0$ the Coriolis force appears, now the third vector component of velocity w influences the motion dramatically, the fluid is "stirred". In this case the following theorem holds:

Theorem. *Let* $(u_0,v_0,w_0)\in C^{1+\theta}(\overline{\Omega})$, $\frac{\partial}{\partial x}u_0+\frac{\partial}{\partial y}v_0=0$ *and* $(u_0n_x+v_0n_y)|_{\partial\Omega}=0$, *where* $0<\theta<1$. *Then a classical solution to the system* (1)–(7) *exists on time interval* $[0,T]$, *where* T *depends on angular velocity of rotation* λ *through the equation*

$$\left(\lambda+\lambda^2\right)T\max\{T,1\}\,\mathrm{e}^{L_1T}=L_2,$$

where constants L_1 *and* L_2 *depend on norms of initial conditions* u_0, v_0, w_0 *in* $C^{1+\theta}(\overline{\Omega})$. *The solution is unique up to an arbitrary function of* t *which may be added to* p.

Let us consider an inviscid incompressible flow in a cylinder of a circular cross section (fluid column). Then $(0,r,0)$ is a stationary solution in a cylindrical coordinates (r,ϕ,z). Mahalov [18] showed that rotation having axis perpendicular to the cylinder symmetry axis has destabilizing effect on the flow. This distortion leads to three-dimensional instabilities. It was shown that rotating fluid columns are unstable to disturbances whose axial wavelengths lie in a band, whose width is proportional to the strength of imposed Coriolis force (λ in this paper). Numerical results were presented on growth rates of mixed modes and on widths of unstable regions. The mathematical result which is proven in this paper is in agreement with physical arguments and numerical simulations presented by Mahalov [18] in the sense that it connects the strength of Coriolis force with the time interval on which a solution is proven to be regular: for weaker Coriolis force the regularity is guaranteed on longer time intervals.

1.2. Notation

Instead of writing (x,y) for the coordinates of Ω we use notation $x=(x_1,x_2)$. A 2-D vector (u,v) is denoted by V. For a vector function V we introduce the matrix

$$\nabla V \stackrel{\text{def}}{=} \begin{pmatrix} u_{x_1} & u_{x_2}\\ v_{x_1} & v_{x_2}\end{pmatrix}.$$

We will work in the space of continuous functions $C^{\varepsilon}(\overline{\Omega})$, $C([0,T];C^{\varepsilon}(\overline{\Omega}))$, $C^{k+\varepsilon}(\overline{Q}_T)$, etc. Norm $\|\cdot\|$ of a scalar function means the usual uniform norm on $C(\overline{\Omega})$, and $\|\cdot\|_{k+\varepsilon}\stackrel{\text{def}}{=}\|\cdot\|_{C([0,T];C^{k+\varepsilon}(\overline{\Omega}))}$. The standard definition of Hölder semi-norms is used:

$$\langle V\rangle_\varepsilon \stackrel{\text{def}}{=} \sup_{\substack{x,y\in\Omega\\ x\neq y}}\frac{|V(x)-V(y)|}{|x-y|^{\varepsilon}}.$$

Norm $\|\cdot\|$ of vector V and of matrix ∇V means $\|u\|+\|v\|$ and $\|u_{x_1}\|+\|u_{x_2}\|+\|v_{x_1}\|+\|v_{x_2}\|$ respectively. The same refers to the seminorms $|\cdot|$, $\langle\cdot\rangle$.

We will use L_i $(i=1,2,\ldots)$ to denote continuously differentiable monotonically increasing functions that are positive at 0. In our estimates, these functions define dependence of constants on the parameters of our problem.

For reader's convenience, we state several lemmas from [17] which will be used in the this paper. In these cases, references to the corresponding lemmas are provided in parentheses.

Finally, all assumptions of the above stated theorem are assumed to be satisfied throughout the paper.

2. Outline of the proof

After applying the "2-D" rot ($\operatorname{curl} V \stackrel{\text{def}}{=} v_{x_1} - u_{x_2}$, a scalar function) to Eqs. (1), (2), we obtain

$$\zeta_t + V \cdot \nabla\zeta = -\lambda w_{x_2}, \tag{8}$$

$$\zeta = \operatorname{curl} V, \tag{9}$$

$$w_t + V \cdot \nabla w = -\lambda u. \tag{10}$$

Now let us suppose the velocity field, $V(x,t)$, is known and sufficiently smooth. Then we can solve the system (8)–(10). First, we solve the differential equation

$$\frac{\mathrm{d}X}{\mathrm{d}t} = V(X,t). \tag{11}$$

We denote its solution with the initial condition $X(s) = x_0$ by $X = X(t; s, x_0)$. Then the solution to (10) is $w(x,t) = w_0(X(0;t,x)) - \lambda \int_0^t u(X(s;t,x),s)\,\mathrm{d}s$. We can see that $u(X(s;t,x),s) = \frac{\mathrm{d}}{\mathrm{d}s} X^{(1)}(s;t,x)$ by (11), where $X^{(1)}$ is the first component of X. Therefore (10) can be rewritten as

$$w(x,t) = w_0\big(X(0;t,x)\big) + \lambda\big(X^{(1)}(0;t,x) - x_1\big), \tag{12}$$

and (8), (9) as:

$$\zeta_t + V \cdot \nabla\zeta = -\lambda \nabla w_0\big(X(0;t,x)\big) \cdot X_{x_2}(0;t,x) - \lambda^2 X^{(1)}_{x_2}(0;t,x), \tag{13}$$

$$\zeta = \operatorname{curl} V. \tag{14}$$

We solve the system (13), (14) as follows. We take a scalar function $\varphi(x,t)$ from some set of functions S and construct V such that for all t we have

$$\operatorname{curl} V = \varphi, \quad \operatorname{div} V = 0, \quad V \cdot n|_{\partial\Omega} = 0. \tag{15}$$

With this V we solve (13) for ζ with initial condition $\zeta|_{t=0} = a \stackrel{\text{def}}{=} \operatorname{curl} V_0$:

$$\zeta(x,t) = a\big(X(0;t,x)\big) + \lambda \int_0^t \nabla w_0\big(X\big(0;s,X(s;t,x)\big)\big) \cdot X_{x_2}\big(0;s,X(s;t,x)\big)\,\mathrm{d}s + \lambda^2 \int_0^t X^{(1)}_{x_2}\big(0;s,X(s;t,x)\big)\,\mathrm{d}s.$$

Using $X(0;s,X(s;t,x)) = X(0;t,x)$ we get

$$\zeta(x,t) = a\big(X(0;t,x)\big) + \lambda \nabla w_0\big(X(0;t,x)\big) \cdot \int_0^t X_{x_2}\big(0;s,X(s;t,x)\big)\,\mathrm{d}s + \lambda^2 \int_0^t X^{(1)}_{x_2}\big(0;s,X(s;t,x)\big)\,\mathrm{d}s \stackrel{\text{def}}{=} \zeta_1 + \zeta_2 + \zeta_3, \tag{16}$$

where we implicitly introduce notations for the three terms of ζ in the expression.

So, the central idea of the proof is as follows. For every $\varphi \in S$ we construct ζ, i.e., we construct a map $\zeta = F(\varphi)$. Now, we choose S so that F maps S into itself and the conditions of Schauder's fixed point theorem hold. If $\varphi = \zeta = F(\varphi)$ is a fixed point, then associated functions u, v, w will be the solution to (1)–(7) along with a pressure p, existence of which is guaranteed by a lemma on orthogonal decomposition of $\mathcal{L}_2$ on solenoid and gradient parts.

In Section 3 we prove all lemmas and preliminary estimates for φ from $\bigcup_{0<\varepsilon<1} C([0,T]; C^\varepsilon(\overline{\Omega}))$. We construct the set S and show that the conditions of Schauder's fixed point theorem are satisfied in Section 4. The existence of a solution to (1)–(7) is established in Section 5. Section 6 proves the uniqueness of the solution.

3. Preliminary lemmas and estimates

Let us construct V for given φ. Let $\psi(x,t)$ be a solution to $\triangle\psi = -\varphi$, $\psi|_{\partial\Omega} = 0$ with fixed t. Then, clearly, $V = (\psi_{x_2}, -\psi_{x_1})$ is the desired function (i.e., which satisfies Eq. (15)).

Lemma 1. *The mapping of $\varphi \to V = (\psi_{x_2}, -\psi_{x_1})$ is continuous from $C([0,T]; C^\theta(\overline{\Omega}))$ to $C([0,T]; C^{1+\theta}(\overline{\Omega}))$:*

$$\|V\|_{1+\theta} \leqslant K\|\varphi\|_\theta.$$

Proof. The conclusion follows from a well-known fact that the mapping of $\varphi \to \psi$ is continuous from $C^\theta(\overline{\Omega})$ to $C^{2+\theta}(\overline{\Omega})$ for fixed t (see [19]), where the constant K does not depend on φ. □

Lemma 2. (Lemmas 2.1 and 2.2; [17]) *Let V be from Lemma* 1. *Then differential equation* (11) *has a unique solution $X = X(t;s,x)$, $t \in [0,T]$ for any initial condition $X(s) = x$, where $s \in [0,T]$ and $x \in \overline{\Omega}$. $X(t;s,x)$ is continuously differentiable as a function of the three variables. For fixed t and s, $X(t;s,x)$ is a one-to-one map of $\overline{\Omega}$ onto itself, where $\partial\Omega$ is mapped onto itself. The Jacobian of this map equals* 1. *$X(s;s,x)$ is the identity map, and $X(s;t,x)$ is the inverse of $X(t;s,x)$.*

Lemmas 1 and 2 justify the calculations in Section 2.

Lemma 3. (Lemma 2.6; [17]) *Let $X(t;s,x)$ be from Lemma* 2. *Then*

$$\big|X(t;s,x) - X(\bar t;\bar s,\bar x)\big| \leqslant L\big(\|\varphi\|\big)\big(|t-\bar t|^\alpha + |s-\bar s|^\alpha + |x-\bar x|^\alpha\big),$$

where $\alpha^{-1} = L(\|\varphi\|)$.

Further we need an estimate of the change of $X(t;s,x)$ and $\nabla X(t;s,x)$. Let us apply ∇ to (11):

$$\frac{\mathrm d}{\mathrm dt}\big(\nabla X(t;s,x)\big) = \nabla V \cdot \nabla X(t;s,x).$$

Solving this equation for the function $\nabla X(t;s,x)$, we get

$$\nabla X(t;s,x) = \exp\left\{\int_s^t \nabla V\big(X(\tau;s,x),\tau\big)\,\mathrm d\tau\right\}. \tag{17}$$

From (17), in particular, we have

$$\big|X(t;s,x) - X(t;s,y)\big| \leqslant \mathrm e^{\|\nabla V\|T}|x-y|.$$

Lemma 4. *Let V be from Lemma* 1 *and $X(t;s,x)$ be from Lemma* 2. *Then*

(i) $|\nabla X(t;s,x) - \nabla X(\bar t;\bar s,\bar x)| \leqslant \|\nabla V\|_\theta \mathrm e^{4\|\nabla V\|T}(T|x-y|^\theta + |t-\bar t| + |s-\bar s|)$;
(ii) $|X(t;s,x) - X(\bar t;s,x)| \leqslant \|V\||t-\bar t|$;
(iii) $|X(t;s,x) - X(t;\bar s,x)| \leqslant \|V\|\,\mathrm e^{\|\nabla V\|T}|s-\bar s|$.

Proof. (i) It is enough to prove the estimate for the next two cases:

(a) $t = \bar t$, $s = \bar s$, (b) $x = \bar x$.

(a) By using the inequality $\|\mathrm e^{A+B} - \mathrm e^A\| \leqslant \|B\|\,\mathrm e^{\|A\|+\|B\|}$, where A and B are matrixes ([20], p. 24), and (17), we obtain the following sequence of inequalities, which gives the desired result:

$$\big|\nabla X(t;s,x) - \nabla X(t;s,y)\big| \leqslant \mathrm e^{3\|\nabla V\||t-s|}\int_s^t \big|\nabla V\big(X(\tau;s,x),\tau\big) - \nabla V\big(X(\tau;s,y),\tau\big)\big|\,\mathrm d\tau$$

$$\leqslant e^{3\|\nabla V\|T}\langle\nabla V\rangle_\theta\int\limits_s^t\big|X(\tau;s,x)-X(\tau;s,y)\big|^\theta\,d\tau$$
$$\leqslant e^{3\|\nabla V\|T}\langle\nabla V\rangle_\theta\|\nabla X\|^\theta\int\limits_s^t|x-y|^\theta\,d\tau$$
$$\leqslant e^{(3+\theta)\|\nabla V\|T}\langle\nabla V\rangle_\theta T|x-y|^\theta.$$

(b) This result follows from the inequality

$$\big|\nabla X(t;s,x)-\nabla X(\bar t;\bar s,x)\big|\leqslant e^{3\|\nabla V\||t-s|}\left\{\int\limits_t^{\bar t}\big\|\nabla V\big(X(\tau;s,x),\tau\big)\big\|\,d\tau+\int\limits_s^{\bar s}\big\|\nabla V\big(X(\tau;s,x),\tau\big)\big\|\,d\tau\right\}$$
$$\leqslant e^{3\|\nabla V\|T}\|\nabla V\|\big\{|t-\bar t|+|s-\bar s|\big\}.$$

(ii) We have

$$\big|X(\bar t;s,x)-X(t;s,x)\big|\leqslant\left|\int\limits_t^{\bar t}V\big(X(\tau;s,x),\tau\big)\,d\tau\right|\leqslant\|V\||s-\bar s|.$$

(iii) Set $z=X(t;s,x)$, $\bar z=X(t;\bar s,x)$ and $x'=X(s;\bar s,x)$. Then we have $x'=X(s;t,\bar z)$ and $\bar z=X(t;s,x')$. First, from (ii) we have the following inequality

$$|x-x'|=\big|X(\bar s;\bar s,x)-X(s;\bar s,x)\big|\leqslant\|V\||s-\bar s|.$$

Next, by (17) we have

$$|z-\bar z|=\big|X(t;s,x)-X(t;s,x')\big|\leqslant e^{\|\nabla V\|T}|x-x'|.$$

By combining these two results we get inequality (iii). □

Lemma 5. *Let α be as in Lemma* 3 *and $\delta=\alpha\theta$. Then for $\zeta=F(\varphi)$ the following inequalities hold*:

$$\|\zeta\|\leqslant\|a\|+\lambda T\,e^{\|\nabla V\|T}\big(\|\nabla w_0\|+\lambda\big),\tag{18}$$
$$\big|\zeta(x,t)-\zeta(y,t)\big|\leqslant\big\{\langle a\rangle_\theta L^\theta\big(\|\varphi\|\big)+\big[\lambda T+(\lambda T)^2\big]\big(1+\|\nabla w_0\|_\theta\big)\big(1+\|\nabla V\|_\theta\big)e^{5\|\nabla V\|T}\big\}|x-y|^\delta,\tag{19}$$
$$\big|\zeta(x,t)-\zeta(x,s)\big|\leqslant L_3\big(\langle a\rangle_\theta,\|\nabla w_0\|_\theta,\|V\|_1,T,\lambda\big)|t-s|^\theta.\tag{20}$$

Proof. The inequality (18) is a consequence of (16) and (17). To prove the rest of the inequalities we estimate ζ_1, ζ_2 and ζ_3 separately for convenience.

By Hölder continuity of a we have

$$\big|\zeta_1(x,t)-\zeta_1(y,t)\big|\leqslant\langle a\rangle_\theta L^\theta\big(\|\varphi\|\big)|x-y|^\delta,\tag{21}$$
$$\big|\zeta_1(x,t)-\zeta_1(x,s)\big|\leqslant\langle a\rangle_\theta\big(\|V\|\,e^{\|\nabla V\|T}|t-s|\big)^\theta,\tag{22}$$

where we used Lemma 3 for the first and Lemma 4(iii) for the second inequality.

To estimate $|\zeta_2(x,t)-\zeta_2(y,t)|$ we add and subtract

$$\lambda\nabla w_0\big(X(0;t,x)\big)\cdot\int\limits_0^t X_{x_2}\big(0;s,X(s;t,y)\big)\,ds$$

and use (17) and Lemma 4(i):

$$\begin{aligned}\left|\zeta_2(x,t)-\zeta_2(y,t)\right| &\leqslant \lambda\left|\nabla w_0\big(X(0;t,x)\big)\right|\int_0^t\left|X_{x_2}\big(0;s,X(s;t,x)\big)-X_{x_2}\big(0;s,X(s;t,y)\big)\right|\mathrm{d}s\\ &\quad+\lambda\left|\nabla w_0\big(X(0;t,x)\big)-\nabla w_0\big(X(0;t,y)\big)\right|\int_0^t\left|X_{x_2}\big(0;s,X(s;t,y)\big)\right|\mathrm{d}s\\ &\leqslant \lambda T\|\nabla w_0\|\,\mathrm{e}^{4\|\nabla V\|T}\|\nabla V\|_\theta\int_0^t\left|X(s;t,x)-X(s;t,y)\right|^\theta\mathrm{d}s\\ &\quad+\lambda T\,\mathrm{e}^{\|\nabla V\|T}\langle\nabla w_0\rangle_\theta\left|X(0;t,x)-X(0;t,y)\right|^\theta\\ &\leqslant \lambda T\big\{\|\nabla w_0\|\,\mathrm{e}^{4\|\nabla V\|T}\|\nabla V\|_\theta\,\mathrm{e}^{\theta\|\nabla V\|T}T+\mathrm{e}^{(1+\delta)\|\nabla V\|T}\langle\nabla w_0\rangle_\theta\big\}|x-y|^\theta.\end{aligned}$$

Hence

$$\left|\zeta_2(x,t)-\zeta_2(y,t)\right|\leqslant\lambda T\|\nabla w_0\|_\theta\,\mathrm{e}^{5\|\nabla V\|T}\big(\|V\|_\theta+1\big)|x-y|^\delta. \tag{23}$$

Similar consideration, plus Lemmas 3 and 4(iii), gives us the estimate

$$\begin{aligned}\left|\zeta_2(x,t)-\zeta_2(x,s)\right| &\leqslant \lambda\left|\nabla w_0\big(X(0;t,x)\big)\right|\int_s^t\left|X_{x_2}\big(0;\tau,X(\tau;t,x)\big)\right|\mathrm{d}\tau\\ &\quad+\lambda\left|\nabla w_0\big(X(0;s,x)\big)-\nabla w_0\big(X(0;t,x)\big)\right|\int_0^s\left|X_{x_2}\big(0;\tau,X(\tau;s,x)\big)\right|\mathrm{d}\tau\\ &\leqslant \lambda\|\nabla w_0\|\,\mathrm{e}^{\|\nabla V\|T}|t-s|+\lambda T\,\mathrm{e}^{\|\nabla V\|T}\langle\nabla w_0\rangle_\theta\|V\|^\theta\,\mathrm{e}^{\theta\|V\|T}|t-s|^\theta\end{aligned}$$

or

$$\left|\zeta_2(x,t)-\zeta_2(x,s)\right|\leqslant\lambda L_4\big(\|\nabla w_0\|_\theta,\|\nabla V\|,T\big)|t-s|^\theta. \tag{24}$$

Finally, by using Lemma 4(i) and (17) again, we get the last two inequalities

$$\begin{aligned}\left|\zeta_3(x,t)-\zeta_3(y,t)\right| &\leqslant \lambda^2\int_0^t\left|X_{x_2}\big(0;s,X(s;t,x)\big)-X_{x_2}\big(0;s,X(s;t,y)\big)\right|\mathrm{d}s\\ &\leqslant \lambda^2T\,\mathrm{e}^{4\|\nabla V\|T}\|\nabla V\|_\theta\int_0^t\left|X(s;t,x)-X(s;t,y)\right|^\theta\mathrm{d}s\\ &\leqslant (\lambda T)^2\,\mathrm{e}^{4+\theta\|\nabla V\|T}\langle\nabla V\rangle_\theta|x-y|^\theta;\end{aligned} \tag{25}$$

$$\left|\zeta_3(x,t)-\zeta_3(x,s)\right|\leqslant\lambda^2\int_s^t\left|X_{x_2}\big(0;\tau,X(\tau;t,x)\big)\right|\mathrm{d}\tau\leqslant\lambda^2\,\mathrm{e}^{\|\nabla V\|T}|t-s|. \tag{26}$$

The estimate (19) follows from the inequalities (21), (23), and (25) taking into account that $\delta=\alpha\theta\leqslant\theta$. The estimate (20) follows from the inequalities (22), (24) and (26). □

4. Application of Schauder's theorem

Let us define the set $S_{A,B}$ as the set of functions $\varphi=\varphi(x,t)$ such that $\|\varphi\|\leqslant A$ and

$$\|\varphi\|_\delta\leqslant\frac{B}{K},\qquad\left|\varphi(x,t)-\varphi(x,s)\right|\leqslant L_3\big(\langle a\rangle_\theta,\|\nabla w_0\|_\theta,B,T,\lambda\big)|t-s|^\theta, \tag{27}$$

where K is the constant from Lemma 1 and $\delta = \theta/L(A)$. It is clear that $S_{A,B} \subset S'$. Moreover, as we will see, for some choice of A and B and some condition on λ, T, the set $S_{A,B}$ has the necessary for application of Schauder's theorem properties, i.e., it is convex, compact and F maps it into itself continuously.

Lemma 6. *For any a and w_0 there exist such positive A, B that F maps $S_{A,B}$ into itself for any λ and T, such they satisfy*

$$\big(\lambda+\lambda^2\big)T\max\{T,1\}\,\mathrm{e}^{L_5(\|a\|,\langle a\rangle_\theta,\|\nabla w_0\|)T} \leqslant \frac{1}{L_6(\|a\|,\langle a\rangle_\theta,\|\nabla w_0\|_\theta)}. \tag{28}$$

Proof. Let A and B be a solution to the system

$$A = \|a\| + \lambda T\,\mathrm{e}^{BT}\big(\|\nabla w_0\|+\lambda\big), \tag{29}$$
$$B = K\big\{\langle a\rangle_\theta L^\theta(A) + N + A\big\}, \tag{30}$$

where N is some positive number. A solution to this system exists for some choice of λ and T. More precisely, rewrite the system (29), (30) as

$$A = f_1(B), \tag{31}$$
$$B = f_2(A). \tag{32}$$

The derivative of the composition $f(x) \stackrel{\text{def}}{=} f_1(f_2(x))$ equals

$$K\lambda T^2\big(\|\nabla w_0\|+\lambda\big)\big(\langle a\rangle_\theta\theta L^{\theta-1}(x)L'(x)+1\big)\,\mathrm{e}^{TK\{\langle a\rangle_\theta L^\theta(x)+N+x\}} \tag{33}$$

and for sufficiently small λ and T will be less then $1-\varepsilon$ for $x \in [0, \frac{f(0)}{\varepsilon}]$, where ε is some positive fixed real number. This guarantees the existence of a solution to the equation $x = f(x)$, because

$$f(0) > 0 \quad \text{and} \quad f\left(\frac{f(0)}{\varepsilon}\right) \leqslant f(0) + (1-\varepsilon)\frac{f(0)}{\varepsilon} = \frac{f(0)}{\varepsilon}.$$

Clearly, a solution A of this equation and its associated B will be a solution to the system (29), (30). It is important here to point out the fact that either λ or T could be chosen in an arbitrary way, and the other constants have to be taken so that the condition above holds. Thus, let us fix λ and choose $T_1(\lambda)$ to guarantee the existence of A and B.

Now we let $T_2 = T_2(\lambda)$ be such that the following inequality holds

$$\big[\lambda T + (\lambda T)^2\big]\big(1+\|\nabla w_0\|_\theta\big)(1+B)\,\mathrm{e}^{5BT} \leqslant N. \tag{34}$$

Taking into account that for $\varphi \in S_{A,B}$ by Lemma 1 we have $\|V\|_{1+\theta} \leqslant B$, for chosen A, B and arbitrary λ and $T \leqslant \min\{T_1(\lambda), T_2(\lambda)\}$, we obtain the following estimates for ζ:

by (18) $\quad \|\zeta\| \leqslant \|a\| + \lambda T\,\mathrm{e}^{BT}\big(\|\nabla w_0\|+\lambda\big) = A;$

by (19) $\quad \big|\zeta(x,t)-\zeta(y,t)\big| \leqslant \big\{\langle a\rangle_\theta L^\theta(A)+N\big\}|x-y|^\delta \leqslant \left(\frac{B}{K}-A\right)|x-y|^\delta, \quad \text{i.e. } \|\zeta\|_\delta \leqslant \frac{B}{K};$

by (20) $\quad \big|\zeta(x,t)-\zeta(x,s)\big| \leqslant L_3\big(\langle a\rangle_\theta, \|\nabla w_0\|_\theta, B, T, \lambda\big)|t-s|^\theta.$

These estimates show us that F maps $S_{A,B}$ into itself.

T–λ dependence (28) easily follows from (33) and (34) if we rewrite these inequalities as

$$\lambda(1+\lambda)T^2\,\mathrm{e}^{L_7(\|a\|,\langle a\rangle_\theta)T} \leqslant \frac{1-\varepsilon}{L_8(\|\nabla w_0\|,\langle a\rangle_\theta)}$$

and

$$\lambda T + (\lambda T)^2\,\mathrm{e}^{5BT} \leqslant \frac{N}{L_9(\|\nabla w_0\|_\theta, B)},$$

where solutions of the system of Eqs. (29) and (30) A and B depend on $\|a\|$, $\langle a\rangle_\theta$, and $\|\nabla w_0\|$ only. □

Lemma 7. *Let $0 < \varepsilon < \beta < 1$, $f_n \in C^\beta(\overline{\Omega})$ be uniformly bounded in $C^\beta(\overline{\Omega})$, $\|f_n - f_0\| \to 0$ for $n \to \infty$. Let X_n be from Lemma 2 and uniformly bounded in $C^1(\overline{\Omega})$, $\|X_n - X_0\| \to 0$, for $n \to \infty$. Then*

$$\|f_n(X_n) - f_0(X_0)\|_{C^\varepsilon(\overline{\Omega})} \to 0.$$

Proof. We have

$$|f_n(X_n(x)) - f_n(X_n(y))| \leqslant \langle f_n\rangle_\beta |X_n(x) - X_n(y)|^\beta \leqslant \langle f_n\rangle_\beta \|X_n\|_{C^1}^\beta |x-y|^\beta,$$

which gives $\|f_n(X_n)\|_{C^\beta} \leqslant \|f_n\| + \langle f_n\rangle_\beta \|X_n\|_{C^1}^\beta$. Consequently $f_n(X_n)$ are bounded in $C^\beta(\overline{\Omega})$. Then by using the theorem on compactness of inclusion $C^\beta(\overline{\Omega})$ in $C^\varepsilon(\overline{\Omega})$ (see, e.g., [19]) we obtain that there exists a subsequence $f_{n_k}(X_{n_k})$ of the sequence $f_n(X_n)$ such that $f_{n_k}(X_{n_k})$ converges in $C^\varepsilon(\overline{\Omega})$: $f_{n_k}(X_{n_k}) \to \psi \in C^\varepsilon$. Let us show that $\psi = f_0(X_0)$:

$$\begin{aligned}\|\psi - f_0(X_0)\| &\leqslant \|\psi - f_n(X_n)\| + \|f_n(X_n) - f_n(X_0)\| + \|f_n(X_0) - f_0(X_0)\| \\ &\leqslant \|\psi - f_n(X_n)\| + \langle f_n\rangle_\beta \|X_n - X_0\|^\beta + \|f_n - f_0\|.\end{aligned}$$

The right-hand side goes to zero as $n \to \infty$. At the same time the left-hand side does not depend on n, thus we have $\psi = f_0(X_0)$.

We showed that any convergent subsequence $f_n(X_n)$ in $C^\varepsilon(\overline{\Omega})$ converges to $f_0(X_0)$, and it, clearly, means convergence of $f_n(X_n)$ to $f_0(X_0)$ in $C^\varepsilon(\overline{\Omega})$. □

Lemma 8. *Let $0 < \varepsilon < \delta$. Then a map F is continuous on $S_{A,B}$ in the topology $C([0,T]; C^\varepsilon(\overline{\Omega}))$.*

Proof. Let $\varphi_n, \varphi \in S_{A,B}$ and $\|\varphi_n - \varphi\|_\varepsilon \to 0$ for $n \to \infty$. Let V_n, X^n, V, X are defined as earlier for φ_n and φ respectively. Then with a help of Lemma 1 we have $\|V_n - V\|_{1+\varepsilon} \to 0$. Therefore $X^n \to X$ uniformly with respect to t, s, x because of a well-known theorem on dependence of a solution to an ordinary differential equations on a right-hand side. By (17) we see that ∇X^n are uniformly bounded in $C([0,T]^2; C^\delta(\overline{\Omega}))$. Next, from Lemma 7 we have

$$\|\nabla V^n(X^n(t;s,x),t) - \nabla V(X(t;s,x),t)\|_{C^\varepsilon(\overline{\Omega})} \to 0, \quad t, s \in [0,T],$$

which gives (using the matrix inequality from the proof of Lemma 4(i), case (a))

$$\|\nabla X^n - \nabla X\|_{C^\varepsilon(\overline{\Omega})} \leqslant \mathrm{e}^{3BT} \int_s^t \|\nabla V^n(X^n(\tau;s,x),\tau) - \nabla V(X(\tau;s,x),\tau)\|_{C^\varepsilon(\overline{\Omega})}\,\mathrm{d}\tau \to 0.$$

Consequently, using Lemma 7 we get

$$\|X^n_{x_2}(0;s,X^n(s;t,x)) - X_{x_2}(0;s,X(s;t,x))\|_{C^\varepsilon(\overline{\Omega})} \to 0, \quad t, s \in [0,T].$$

Now we recall that ∇X^n and X^n are uniformly Lipshitz with respect to t and s (from Lemma 4 (ii) and (iii)). It means $\nabla X^n(0;s,X^n(s;t,x))$ are uniformly Lipshitz with respect to t and s. Then

$$\|X^n_{x_2}(0;s,X^n(s;t,x)) - X_{x_2}(0;s,X(s;t,x))\|_{C^\varepsilon(\overline{\Omega})} \to 0$$

uniformly with respect to t and s. In other words, $\zeta_3^n \to \zeta_3$ uniformly with respect to t. This, with the help of Lemma 7 for $\nabla w_0(X^n)$, gives convergence of ζ_2^n to ζ_2. The same lemma for $a(X^n)$ shows convergence of $\zeta_1^n \to \zeta_1$ in $C([0,T]; C^\varepsilon(\overline{\Omega}))$. □

Convexity of the set $S_{A,B}$ is ostensible. To use the fixed point theorem we have only to show its compactness in the Banach space $C([0,T]; C^\varepsilon(\overline{\Omega}))$.

Lemma 9. *Let δ be from Lemma 5 and ε be such that $0 < \varepsilon < \delta$. Then $S_{A,B}$ is compact in $C([0,T]; C^\varepsilon(\overline{\Omega}))$.*

Proof. We have $\varepsilon < \delta$ and $S_{A,B}$ is bounded in $C^\delta(\overline{Q}_T)$. Recalling the fact on compact inclusion $C^{\delta'}(\overline{Q}_T) \hookrightarrow C^\delta(\overline{Q}_T)$ for $\delta' > \delta$ (see [19]), we conclude that $S_{A,B}$ is a compact set in $C^\varepsilon(\overline{Q}_T)$ and, therefore, in $C([0,T]; C^\varepsilon(\overline{\Omega}))$. □

5. Existence

Let $\varphi \in S_{A,B}$ be a fixed point of F, $\varphi = \zeta = F(\varphi)$, and let $X(t;s,x)$, $V(x,t)$, $w(x,t)$ be functions according to this φ. Because $X(t;s,x)$ is a continuously differentiable function, and $w_0 \in C^{1+\theta}(\overline{\Omega})$, from (12) we have $w \in C^1(\overline{Q}_T)$. From Lemma 4(i) we have

$$\left|\nabla X(t;s,x) - \nabla X(\bar t;\bar s,\bar x)\right| \leqslant B\,\mathrm{e}^{4BT}\left(T|x-y|^\delta + |t-\bar t| + |s-\bar s|\right)$$

and, consequently, $X(t;s,x) \in C^{1+\delta,\delta}([0,T]^2 \times \overline{\Omega})$. Then from (12), we conclude that $w \in C^{1+\delta,0}(\overline{Q}_T)$. Now all the conditions of the theorem on existence of a classical solution to the 2-D Euler equations in [17] hold for $b = w$. Our map F is F from [17] and, consequently, the following lemma holds:

Lemma 10. (Lemma 3.1, Lemma 3.2, Lemma 3.3; [17])

(i) $\nabla V \in C^{1+\delta',\delta'}(\overline{Q}_T)$ *for any* $\delta' < \delta$;
(ii) V_t *exists and is from* $C(\overline{Q}_T)$;
(iii) *There exists a scalar function* $p \in C^{1,0}(\overline{Q}_T)$ *such that* $\{V, p\}$ *is a solution to* (1), (2), (4), (5), (7).

6. Uniqueness

Let us suppose there is another solution $(\widetilde V, \widetilde w, \tilde p)$ of Eqs. (1)–(7), which satisfies the conditions of the theorem. After subtracting the equations with the solution (V, w, p) from the equations with the solution $(\widetilde V, \widetilde w, \tilde p)$ and denoting $U = (u_1, u_2) = \widetilde V - V$, $W = \widetilde w - w$, $q = \tilde p - p$, we have:

$$U_t + \big(\widetilde V\nabla\big)U + (U\nabla)V = -\nabla q + \lambda(W, 0), \tag{35}$$

$$W_t + \widetilde V\nabla W = -\lambda u_1 - U\nabla w. \tag{36}$$

We take the scalar product of (35) with U. Then, noting that $(U, (\widetilde V\nabla)U) = -((\widetilde V\nabla)U, U) = 0$ and $(U, \nabla q) = 0$ we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(U,U) + \big(U, (U\nabla)V\big) = \lambda\int_\Omega u_1 W\,\mathrm{d}x. \tag{37}$$

By (36) we have

$$W(x,t) = -\lambda\int_0^t u_1\big(\widetilde X(s;t,x), s\big)\,\mathrm{d}s - \int_0^t U\big(\widetilde X(s;t,x), s\big)\nabla w\big(\widetilde X(s;t,x), s\big)\,\mathrm{d}s.$$

Let us denote these two terms by W_1 and W_2 respectively. By using this notation for W we obtain an estimate for the right part of (37):

$$\begin{aligned}
\left|\int_\Omega W_1 u_1\,\mathrm{d}x\right| &\leqslant \lambda^2\left|\int_0^t\int_\Omega u\big(\widetilde X(s;t,x), s\big)u(x,t)\,\mathrm{d}x\,\mathrm{d}s\right| \leqslant \frac{\lambda^2}{2}\int_0^t\left\{\int_\Omega U^2(y,s)\,\mathrm{d}y + \int_\Omega U^2(x,t)\,\mathrm{d}x\right\}\mathrm{d}s\\
&\leqslant \frac{\lambda^2 T}{2}(U,U) + \frac{\lambda^2 t}{2}\max_{0\leqslant\tau\leqslant t}(U,U),\\
\left|\int_\Omega W_2 u_1\,\mathrm{d}x\right| &\leqslant \lambda\left|\int_0^t\int_\Omega U\big(\widetilde X(s;t,x), s\big)\nabla w\big(\widetilde X(s;t,x), s\big)u_1(x,t)\,\mathrm{d}s\right|\\
&\leqslant \frac{\lambda}{2}\int_0^t\left\{\int_\Omega\big(U(y,s)\nabla w(y,s)\big)^2\,\mathrm{d}y + \int_\Omega U^2(x,t)\,\mathrm{d}x\right\}\mathrm{d}s.
\end{aligned} \tag{38}$$

Here we used the substitution $y = \widetilde{X}(s; t, x)$ and the fact that its Jacobian equals 1 from Lemma 2. It is known that $w \in C^1(\overline{Q}_T)$, therefore, the last inequality can be rewritten as:

$$\left| \int_\Omega W_2 u_1 \, dx \right| \leqslant \frac{\lambda T}{2} K_1 \max_{0 \leqslant \tau \leqslant t} (U, U) + \frac{\lambda T}{2} (U, U). \tag{39}$$

We have $V \in C^1(\overline{Q}_T)$, therefore (37), (38) and (39) gives

$$(U, U)_t \leqslant K_2(U, U) + \frac{\lambda + \lambda^2}{2} T\big((U, U) + \max_{0 \leqslant \tau \leqslant t} (U, U)\big).$$

Evaluating the integral $\int_s^t$ of this inequality and denoting $(U, U)(t) = \varphi(t)$, we obtain

$$\varphi(t) \leqslant \varphi(s) + K_3 \int_s^t \varphi(\tau) \, d\tau + K_4 |t - s| \max_{0 \leqslant \tau \leqslant t} \varphi(\tau), \tag{40}$$

where

$$K_3 = K_2 + \frac{\lambda + \lambda^2}{2} T \quad \text{and} \quad K_4 = \frac{\lambda + \lambda^2}{2} T.$$

Let us define $M \stackrel{\text{def}}{=} \max_{s \leqslant \tau \leqslant t} \varphi(\tau)$ and take $s = 0$ (it is known that $\varphi(0) = 0$) and $t = t_1 \stackrel{\text{def}}{=} \frac{1}{K_3 + K_4 + 1}$. Then (40) can be rewritten (because of monotonicity of the right part of the inequality (40) we can substitute $\varphi(t)$ by M):

$$M \leqslant (K_3 + K_4) t_1 M.$$

In this case, clearly, $M = 0$, i.e., $\varphi(t) = 0$ on $[0, t_0]$. Constants K_3, K_4 do not depend on t, so, repeating the arguments for $s = t_1$ and $t = 2t_1$, etc, we can conclude that $\varphi(t) \equiv 0$, $t \in [0, T]$.

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